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Applications of complex analysis to precession, nutation and aberration

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ABSTRACT

Points on the surface of a sphere can be mapped by stereographic projection to points on the plane of complex numbers. If the points on the sphere are identified with the directions of incoming light rays, then the effect of a Lorentz transformation, a rotation plus a boost, is represented by a bilinear or Möbius transformation applied to points on the complex plane. This procedure allows the effects of the aberration of light, precession and nutation, required for computing the mean and apparent positions of celestial objects, to be accounted for in a common framework and yields expressions that are readily evaluated in practice. The general form of the bilinear transformation representing a pure Lorentz boost is derived. Explicit expressions are given for the bilinear transformations representing aberration, precession and nutation as well as frame bias and transformations to the Celestial Intermediate Reference System. The approach described simplifies, and is an alternative to, the standard matrix methods commonly used to perform coordinate system rotations.

Key words: relativity – celestial mechanics – ephemerides – reference systems.

1 INTRODUCTION

Several methods are available for the practical conversion between spherical coordinate systems (Heard 2006). In the most familiar and widely used of these, a point on the sphere is represented by a vector of unit length in three-dimensional (3D) space. Rotations are then performed by the action of classical rotation matrices. Rotations may also be performed by representing points as the components of a quaternion and appealing to the properties of quaternion multiplication. The quaternion components are related to the Gibbs vector whose direction and magnitude specify the axis and angle of the rotation. This is briefly discussed in connection with IAU precession by Capitaine, Wallace & Chapront (2003). As an alternative to the 3D representation of points on the sphere, their coordinates may be used to form the elements of a 2 × 2 complex matrix with rotations being brought about by 2 × 2 matrix multiplication (Arfken & Weber 1995). This approach arises in the quantum mechanics of spin 1/2 objects. It can be demonstrated that the complex 2 × 2 matrix formalism is general enough to provide an exact representation of Lorentz transformations.

A point on a sphere is fully specified by two coordinates. The 3D, complex 2 × 2 matrix and quaternion representations introduce additional degrees of freedom and must be applied subject to constraints. An alternative method, that avoids this requirement, is to map points on the sphere by stereographic projection on to points in the complex number plane. Rotations of the sphere are then isomorphic to simple bilinear or Möbius transformations and can be used in practice to carry out coordinate transformations. This method has been discussed previously in the context of astronomical applications (Stuart 1984). Moreover, it can be shown (Penrose & Rindler 1984; Das 1993; Bacry 2004) that the approach is sufficiently general as to incorporate the effect of an arbitrary Lorentz transformation on light-like world lines passing through the observer's position or, equivalently, light rays seen projected on the celestial sphere. The upshot is that simple bilinear transformations can be used to exactly represent both coordinate rotations and the aberration of light. As they are merely different aspects of an overall Lorentz transformation, it is natural that they be treated under a common framework.

The use of bilinear transformations to represent rotations and Lorentz boosts offers a number of significant practical advantages. The two coordinates that define a point on the sphere are subsumed into a single complex number leading to expressions of great simplicity, transparency and compactness. Complex numbers are a native feature of most computer languages used in scientific applications, FORTRAN, C++ etc., and the bilinear transformations can be straightforwardly evaluated in practice using a small number of built in arithmetic operations.

Stereographic projection of spherical coordinates on to the complex number plane has proved useful elsewhere. It has been used as a tool to derive spherical trigonometric identities from plane trigonometry (Donnay 1945; Stuart 1984), for calculations in celestial navigation (Stuart 2009) and to treat problems in rotational dynamics with applications to spacecraft control systems (Tsiotras 1996).

A disadvantage of the method is that absence of a redundant degree of freedom necessarily comes at the price of a singularity

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occurring in some direction. In practice this is of little consequence as the singularity can be arranged to be at a point, e.g. the North or South Pole, away from the region of interest. A singularity with the same origins is present in the Gibbs vector and methods for placing it at an arbitrary location have been given by Schaub & Junkins (1996).

In this work, the method of stereographic projection along with bilinear transformations is reviewed. Expressions for the coefficients of the bilinear transformations used to represent the IAU precession quantities (IERS Conventions 2003; Hilton et al. 2006) will be given. In addition, an expression for the bilinear transformation representing a Lorentz boost in terms of the magnitude and direction of the observer's velocity is derived. This results in a formalism incorporating coordinate rotations and the aberration of light into a simple common framework that is straightforwardly applicable in practice.

The transformations discussed here require knowledge only of the direction of an object in some reference frame. Other effects come into play in the computation of an object's apparent position. In particular light travel time and the gravitational deflection of light require information as to the object's distance and are not considered here. It will be assumed that these effects are treated by standard methods (Murray 1981). Algorithms for mean and apparent place computations have been given by Kaplan et al. (1989). The results presented here allow a number of the steps to be replaced by an alternative, more compact, framework.

2 STEREOGRAPHIC PROJECTION AND BILINEAR TRANSFORMATIONS

In complex number theory, stereographic projection is used to establish an isomorphism between points on the complex plane and points lying on a sphere – the Riemann sphere. Consider a point, P, with 3D Cartesian coordinates (X, Y, Z) lying on the surface of a unit sphere centred at the origin and hence satisfying the condition $X^2 + Y^2 + Z^2 = 1$.

Let the point's spherical coordinates be (α, δ) with

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix}. \tag{1}$$

The spherical coordinates could represent right ascension/declination, longitude/latitude or other similar pair. Stereographic projection can be performed in a number of equivalent ways. For the present purposes, the complex number plane is taken to be coincident with the X–Y plane. As shown in Fig. 1, a straight line is drawn from the North Pole, N, of the sphere through the point P to intersect with the complex plane at the complex number,

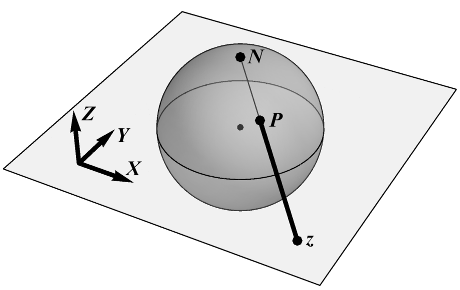

Figure 1. Stereographic projection of the point, P, on the sphere on to the point, z, on the complex number plane.

z, given by

$$z = \tan\left(\frac{\pi}{4} + \frac{\delta}{2}\right) \mathrm{e}^{\mathrm{i}\alpha} = \frac{X + \mathrm{i}Y}{1 - Z}. \tag{2}$$

For points on the sphere lying below the X–Y plane ($Z < 0$) the point of intersection will be inside the sphere. The projection can be inverted using

$$\alpha = \arg z, \qquad \delta = 2\tan^{-1}|z| - \frac{\pi}{2} \tag{3}$$

or in terms of rectangular coordinates

$$X = \frac{2\mathrm{Re}z}{|z|^2+1}, \qquad Y = \frac{2\mathrm{Im}z}{|z|^2+1}, \qquad Z = \frac{|z|^2-1}{|z|^2+1}. \tag{4}$$

A bilinear or Möbius transformation takes the general form

$$\mathrm{T}(z) = \frac{az+b}{cz+d} \tag{5}$$

for complex constants a, b, c and d. The coefficients are often normalized such that $ad - bc = 1$ although this not a requirement when used in the bilinear transformation (5). The inverse transformation is of the same form

$$\mathrm{T}^{-1}(z) = \frac{dz-b}{-cz+a}. \tag{6}$$

It can be shown (Nevanlinna & Paatero 1969) that the effect of a pure rotation of the Riemann sphere is represented by a bilinear transformation

$$\mathrm{T}(z) = \frac{az+b}{-\bar{b}z+\bar{a}}, \tag{7}$$

where the bar denotes complex conjugation. Bilinear transformations of this type will be referred to as bilinear rotations. In this case the normalization condition becomes

$$|a|^2 + |b|^2 = 1 \tag{8}$$

under which a and b may be identified with the Cayley–Klein parameters (Arfken & Weber 1995) whose real and imaginary parts are further simply related to the components of quaternions used to represent rotations. A brief overview is given in Appendix A. Derivations and numerical examples of coordinate system rotations performed using equation (7) can be found in the literature (Stuart 1984).

In the form (2), the stereographic projection exhibits a singularity at the North Pole. The singularity may be placed at the South Pole instead by adopting the transformation

$$z = \tan\left(\frac{\pi}{4} - \frac{\delta}{2}\right) \mathrm{e}^{-\mathrm{i}\alpha} = \frac{X - \mathrm{i}Y}{1 + Z} \tag{9}$$

with inversion formulae

$$\alpha = -\arg z, \qquad \delta = -2\tan^{-1}|z| + \frac{\pi}{2}, \tag{10}$$

$$X = \frac{2\mathrm{Re}z}{1+|z|^2}, \qquad Y = -\frac{2\mathrm{Im}z}{1+|z|^2}, \qquad Z = \frac{1-|z|^2}{1+|z|^2}. \tag{11}$$

In the present work, results will be derived assuming a stereographic projection in the form (2). To adapt them for use in conjunction with the stereographic projection of the form given in (9), the coefficients of the bilinear rotation are switched $a \leftrightarrow d$ and $b \leftrightarrow c$.

General coordinate rotations can often be conveniently constructed from a sequence of elemental rotations. Bilinear rotations

Table 1. The fundamental rotation matrices and their equivalent bilinear rotation coefficients.

Rotation matrix	Bilinear coefficients
$\mathbf{R}_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$	$a = \cos\frac{\phi}{2}$, $b = -\mathrm{i}\sin\frac{\phi}{2}$
$\mathbf{R}_2(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$	$a = \cos\frac{\phi}{2}$, $b = \sin\frac{\phi}{2}$
$\mathbf{R}_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$a = \exp\left(-\mathrm{i}\frac{\phi}{2}\right)$, $b = 0$

can be composed in a similar fashion:

$$\mathrm{T}_2 \circ \mathrm{T}_1(z) = \frac{a_2\left[(a_1 z + b_1)/(-\bar{b}_1 z + \bar{a}_1)\right] + b_2}{-\bar{b}_2\left[(a_1 z + b_1)/(-\bar{b}_1 z + \bar{a}_1)\right] + \bar{a}_2} = \frac{\left(a_1 a_2 - \bar{b}_1 b_2\right) z + \left(b_1 a_2 + \bar{a}_1 b_2\right)}{-\left(\bar{b}_1 \bar{a}_2 + a_1 \bar{b}_2\right) z + \left(\bar{a}_1 \bar{a}_2 - b_1 \bar{b}_2\right)}. \tag{12}$$

To facilitate the derivation of precession quantities, Table 1 lists the fundamental rotation matrices $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ and bilinear rotation coefficients, a and b, for use in equation (7), that perform the equivalent rotation assuming stereographic projection of the form (2). The rotation matrices, given here, act on a column vector, (1), by multiplication on its left.

3 PRECESSION, NUTATION AND FRAME BIAS

The conversion from coordinates specified with respect to some fundamental 'fixed' reference system to those with respect to the true equator and equinox of date requires, among other things, that effects of precession, nutation and any potential systematic frame bias be taken into account. Standard procedures in accordance with IAU resolutions are collected in USNO Circular No. 179 (Kaplan 2005). If the fundamental reference system is the Geocentric Celestial Reference System (GCRS), coordinates are first converted to the J2000.0 system in terms of the mean equator and equinox of J2000.0 via a small frame bias correction realized by a matrix $\mathbf{B}$. Transformations for precession to the mean equator and equinox of date via a matrix, $\mathbf{P}(t)$, and nutation to the true equator and equinox of date via a matrix, $\mathbf{N}(t)$, are then applied. Taken together

$$\boldsymbol{r}_{\mathrm{true}(t)} = \mathbf{N}(t)\mathbf{P}(t)\mathbf{B}\,\boldsymbol{r}_{\mathrm{GCRS}}, \tag{13}$$

where $\boldsymbol{r}_{\mathrm{GCRS}}$ is a position vector in the GCRS and $\boldsymbol{r}_{\mathrm{true}(t)}$ is the equivalent vector given with respect to the true equator and equinox of t. In the following sections T denotes the time in Julian centuries since J2000.0,

$$T = \frac{(t - 245\,1545.0)}{36525}, \tag{14}$$

where t is expressed in Barycentric Dynamical Time (TDB) Julian days.

3.1 Precession

The most general 3D rotation can be specified in terms of three Euler angles. Construction of a rotation matrix from a greater number of parameters is a matter of convenience. The traditional three-parameter form (Newcomb 1894; Lieske 1979) of the precession matrix, $\mathbf{P}(t)$, is

$$\mathbf{P}(t) = \mathbf{R}_3(-z_A)\,\mathbf{R}_2(\theta_A)\,\mathbf{R}_3(-\zeta_A). \tag{15}$$

The coefficients of the equivalent bilinear rotation, (7), are

$$a = \cos\frac{\theta_A}{2}\exp\left(\mathrm{i}\frac{z_A + \zeta_A}{2}\right), \tag{16}$$

$$b = \sin\frac{\theta_A}{2}\exp\left(\mathrm{i}\frac{z_A - \zeta_A}{2}\right) \tag{17}$$

in which the arguments in arcseconds are (Hilton et al. 2006)

$$\begin{aligned}
\frac{1}{2}(z_A - \zeta_A) &= -2.650545 - 0.003023T + 0.3969425T^2 \\
&\quad + 0.00012504T^3 - 0.000011313T^4 \\
&\quad + 0.0000000135T^5, \\
\frac{1}{2}(z_A + \zeta_A) &= 2306.080204T + 0.6957924T^2 \\
&\quad + 0.01814333T^3 - 0.000017284T^4 \\
&\quad - 0.0000003039T^5, \\
\theta_A &= 2004.191903T - 0.4294934T^2 \\
&\quad - 0.04182264T^3 - 0.000007089T^4 \\
&\quad - 0.0000001274T^5.
\end{aligned}$$

The four-parameter representation of $\mathbf{P}(t)$ (Capitaine et al. 2003) provides a clean separation of precession of the equator from precession of the ecliptic allowing precession rate adjustments to be directly applied. This precession matrix is

$$\mathbf{P}(t) = \mathbf{R}_3(\chi_A)\,\mathbf{R}_1(-\omega_A)\,\mathbf{R}_3(-\psi_A)\,\mathbf{R}_1(\epsilon_0), \tag{18}$$

where ϵ_0 is the obliquity of the ecliptic at J2000.0, $\epsilon_0 = 84381.406000$.

The coefficients of the equivalent bilinear rotation are

$$a = \cos\frac{\epsilon_0}{2}\cos\frac{\omega_A}{2}\exp\left(\mathrm{i}\frac{\psi_A - \chi_A}{2}\right) + \sin\frac{\epsilon_0}{2}\sin\frac{\omega_A}{2}\exp\left(-\mathrm{i}\frac{\psi_A + \chi_A}{2}\right), \tag{19}$$

$$b = -\mathrm{i}\sin\frac{\epsilon_0}{2}\cos\frac{\omega_A}{2}\exp\left(\mathrm{i}\frac{\psi_A - \chi_A}{2}\right) + \mathrm{i}\cos\frac{\epsilon_0}{2}\sin\frac{\omega_A}{2}\exp\left(-\mathrm{i}\frac{\psi_A + \chi_A}{2}\right) \tag{20}$$

which can be written, fully in terms of half-sums and half-differences of angles, as

$$\begin{aligned}
a &= \frac{1}{2}\left[\cos\left(\frac{\epsilon_0 + \omega_A}{2}\right) + \cos\left(\frac{\epsilon_0 - \omega_A}{2}\right)\right]\exp\left(\mathrm{i}\frac{\psi_A - \chi_A}{2}\right) \\
&\quad - \frac{1}{2}\left[\cos\left(\frac{\epsilon_0 + \omega_A}{2}\right) - \cos\left(\frac{\epsilon_0 - \omega_A}{2}\right)\right] \\
&\quad \times \exp\left(-\mathrm{i}\frac{\psi_A + \chi_A}{2}\right),
\end{aligned} \tag{21}$$

$$\begin{aligned}
b &= -\frac{\mathrm{i}}{2}\left[\sin\left(\frac{\epsilon_0 + \omega_A}{2}\right) + \sin\left(\frac{\epsilon_0 - \omega_A}{2}\right)\right]\exp\left(\mathrm{i}\frac{\psi_A - \chi_A}{2}\right) \\
&\quad + \frac{\mathrm{i}}{2}\left[\sin\left(\frac{\epsilon_0 + \omega_A}{2}\right) - \sin\left(\frac{\epsilon_0 - \omega_A}{2}\right)\right] \\
&\quad \times \exp\left(-\mathrm{i}\frac{\psi_A + \chi_A}{2}\right).
\end{aligned} \tag{22}$$

The overall factor of 1/2 in expressions for a and b in equations (21) and (22) can be dropped when used in equation (7) as it does not affect the final result. The function arguments in arcseconds are

$$\begin{aligned}\frac{1}{2}(\epsilon_0 - \omega_A) &= -0.012877T + 0.0256312T^2 \\ &\quad - 0.00386252T^3 - 0.000000234T^4 \\ &\quad + 0.0000001669T^5, \\ \frac{1}{2}(\psi_A + \chi_A) &= 2524.518955T - 1.7302181T^2 \\ &\quad - 0.00117621T^3 + 0.000151757T^4 \\ &\quad - 0.0000000756T^5, \\ \frac{1}{2}(\psi_A - \chi_A) &= 2513.962552T + 0.6512112T^2 \\ &\quad + 0.00003576T^3 - 0.000018906T^4 \\ &\quad - 0.0000000196T^5\end{aligned}$$

with $(1/2)(\epsilon_0 + \omega_A) = \epsilon_0 - (1/2)(\epsilon_0 - \omega_A)$. Note that all of the polynomials given explicitly above vanish at $T = 0$.

3.2 Nutation

In transformations (15) and (18) in which nutation is applied separately, the corrections are defined in terms of the mean obliquity of the ecliptic of date, ϵ_A, given by

$$\begin{aligned}\epsilon_A = \epsilon_0 &- 46.836769T - 0.0001831T^2 \\ &+ 0.00200340T^3 - 0.000000576T^4 \\ &- 0.0000000434T^5\end{aligned}$$

along with the luni-solar and planetary nutation in obliquity, $\Delta\epsilon$, and longitude, $\Delta\psi$, computed using a compatible nutation model (Seidelmann 1982; Matthews, Herring & Buffet 2002; Capitaine, Wallace & Chapront 2003).

The general form of the nutation matrix is

$$\mathbf{N}(t) = \mathbf{R}_1(-[\epsilon_A + \Delta\epsilon])\,\mathbf{R}_3(-\Delta\psi)\,\mathbf{R}_1(\epsilon_A) \tag{23}$$

for which the coefficients of the equivalent bilinear rotation are

$$a = \cos\frac{\Delta\epsilon}{2}\cos\frac{\Delta\psi}{2} + \mathrm{i}\cos\left(\epsilon_A + \frac{\Delta\epsilon}{2}\right)\sin\frac{\Delta\psi}{2}, \tag{24}$$

$$b = \sin\left(\epsilon_A + \frac{\Delta\epsilon}{2}\right)\sin\frac{\Delta\psi}{2} + \mathrm{i}\sin\frac{\Delta\epsilon}{2}\cos\frac{\Delta\psi}{2}. \tag{25}$$

3.3 Frame bias

The ICRS reference frame is nominally aligned with the J2000.0 coordinate system, however, a small offset, or frame bias, exists with conversion between the two taking the form

$$\boldsymbol{r}_{\mathrm{mean}(2000)} = \mathbf{B}\boldsymbol{r}_{\mathrm{GCRS}}. \tag{26}$$

The matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \mathbf{R}_1(-\eta_0)\,\mathbf{R}_2(\xi_0)\,\mathbf{R}_3(d\alpha_0), \tag{27}$$

where $d\alpha_0 = -14.6$ mas, $\xi_0 = -16.6170$ mas and $\eta_0 = -6.8192$ mas. The equivalent bilinear rotation has coefficients

$$a = \left[\cos\frac{\eta_0}{2}\cos\frac{\xi_0}{2} - \mathrm{i}\sin\frac{\eta_0}{2}\sin\frac{\xi_0}{2}\right]\exp\left(-\mathrm{i}\frac{d\alpha_0}{2}\right), \tag{28}$$

$$b = \left[\cos\frac{\eta_0}{2}\sin\frac{\xi_0}{2} + \mathrm{i}\sin\frac{\eta_0}{2}\cos\frac{\xi_0}{2}\right]\exp\left(\mathrm{i}\frac{d\alpha_0}{2}\right) \tag{29}$$

which can alternatively be written

$$\begin{aligned}a = &\frac{1}{\sqrt{2}}\cos\left(\frac{\eta_0 - \xi_0}{2}\right)\exp\left(-\mathrm{i}\left[\frac{d\alpha_0}{2} + \frac{\pi}{4}\right]\right) \\ &+ \frac{1}{\sqrt{2}}\cos\left(\frac{\eta_0 + \xi_0}{2}\right)\exp\left(-\mathrm{i}\left[\frac{d\alpha_0}{2} - \frac{\pi}{4}\right]\right),\end{aligned} \tag{30}$$

$$\begin{aligned}b = &\frac{1}{\sqrt{2}}\sin\left(\frac{\eta_0 - \xi_0}{2}\right)\exp\left(\mathrm{i}\left[\frac{d\alpha_0}{2} + \frac{3\pi}{4}\right]\right) \\ &+ \frac{1}{\sqrt{2}}\sin\left(\frac{\eta_0 + \xi_0}{2}\right)\exp\left(\mathrm{i}\left[\frac{d\alpha_0}{2} + \frac{\pi}{4}\right]\right),\end{aligned} \tag{31}$$

where the overall factor of $1/\sqrt{2}$ imposes the unimodularity condition (8) but can be dropped when used in bilinear rotations, (7). Up to second order in the variables $d\alpha_0, \xi_0$ and η_0,

$$a = 1 - \frac{1}{8}(d\alpha_0)^2 - \frac{1}{8}\eta_0^2 - \frac{1}{8}\xi_0^2 - \mathrm{i}\left(\frac{1}{2}d\alpha_0 + \frac{1}{4}\eta_0\xi_0\right), \tag{32}$$

$$b = \frac{1}{2}\xi_0 - \frac{1}{4}d\alpha_0\eta_0 + \mathrm{i}\left(\frac{1}{2}\eta_0 + \frac{1}{4}d\alpha_0\xi_0\right). \tag{33}$$

4 COMBINED ROTATIONS

4.1 Fukushima parametrization

The four-parameter form of the rotation matrix (Williams 1994; Fukushima 2003)

$$\mathbf{P}(t) = \mathbf{R}_1(-\epsilon_A)\,\mathbf{R}_3(-\psi)\,\mathbf{R}_1(\phi)\,\mathbf{R}_3(\gamma) \tag{34}$$

possesses a number of significant practical advantages described below. As the transformation involves four rotation angles, its bilinear rotation coefficients will take a form analogous to equations (19), (20) and equations (21), (22). Here only the former, more compact version, will be given and is

$$\begin{aligned}a = &\cos\frac{\epsilon_A}{2}\cos\frac{\phi}{2}\exp\left(\mathrm{i}\frac{\psi - \gamma}{2}\right) \\ &+ \sin\frac{\epsilon_A}{2}\sin\frac{\phi}{2}\exp\left(-\mathrm{i}\frac{\psi + \gamma}{2}\right),\end{aligned} \tag{35}$$

$$\begin{aligned}b = &\mathrm{i}\sin\frac{\epsilon_A}{2}\cos\frac{\phi}{2}\exp\left(-\mathrm{i}\frac{\psi - \gamma}{2}\right) \\ &- \mathrm{i}\cos\frac{\epsilon_A}{2}\sin\frac{\phi}{2}\exp\left(\mathrm{i}\frac{\psi + \gamma}{2}\right).\end{aligned} \tag{36}$$

For the conversion from J2000.0 mean equator and ecliptic to the mean equator and ecliptic of date,

$$\boldsymbol{r}_{\mathrm{mean}(t)} = \mathbf{P}(t)\boldsymbol{r}_{\mathrm{mean}(2000)}. \tag{37}$$

The angles appearing in equations (35) and (36), in arcseconds, are

$$\begin{aligned}\phi &= \epsilon_0 - 46.811015T + 0.0511269T^2 \\ &\quad + 0.00053289T^3 - 0.000000440T^4 \\ &\quad - 0.0000000176T^5, \\ \frac{1}{2}(\psi - \gamma) &= 2513.962552T + 0.5326066T^2 \\ &\quad + 0.00006358T^3 - 0.000011832T^4 \\ &\quad - 0.0000000204T^5, \\ \frac{1}{2}(\psi + \gamma) &= 2524.518955T + 1.0258110T^2 \\ &\quad - 0.00024880T^3 - 0.000014620T^4 \\ &\quad + 0.0000000056T^5.\end{aligned}$$

The arguments in equations (35) and (36) are expressed in terms of polynomials of which two, ϵ_A and ϕ, are non-vanishing at $T = 0$. The alternative form, analogous to equations (30) and (31), has arguments only one of which, $(\epsilon_A + \phi)/2$, does not vanish at $T = 0$. The arguments may also be constructed so as to incorporate the effects of frame bias for conversion directly from GCRS to coordinates referred to the mean equator and ecliptic of date. This introduces a constant and affects up to the quadratic terms in T. The parametrization (34) has the added advantage that the effect of nutation can be simply included by adding the nutation in obliquity, $\Delta\epsilon$, and longitude, $\Delta\psi$, directly into equation (34), and hence into equations (35) and (36), as $\epsilon_A \to \epsilon_A + \Delta\epsilon$ and $\psi \to \psi + \Delta\psi$. Thus this single matrix form conveniently incorporates frame bias, precession and nutation.

4.2 The Celestial Intermediate Reference System (CIRS)

The net effect of precession, nutation and frame bias, or any coordinate system rotation in general, can be described by the location of the pole of the new coordinate system followed by a rotation about that pole. This forms the basis for CIRS used in the measurement of the Earth's rotation (IERS Conventions 2003). The transformation from GCRS to CIRS takes the usual form

$$\boldsymbol{r}_{\mathrm{CIRS}} = \mathbf{C}(t)\,\boldsymbol{r}_{\mathrm{GCRS}}. \tag{38}$$

The matrix $\mathbf{C}(t)$ is defined in terms of the location Celestial Intermediate Pole (CIP) with coordinates (X, Y, Z) in the GCRS and can be written as

$$\mathbf{C}(t) = \mathbf{R}_3(-s)\begin{pmatrix} 1-\hat{a}X^2 & -\hat{a}XY & -X \\ -\hat{a}XY & 1-\hat{a}Y^2 & -Y \\ X & Y & 1-\hat{a}\left(X^2+Y^2\right) \end{pmatrix}$$

in which $\hat{a} = (1+Z)^{-1}$ and $Z = \sqrt{1-X^2-Y^2}$. The quantity s is an angle of rotation about the CIP called the Celestial Intermediate Origin (CIO) locator. This matrix can simultaneously encapsulate the effects of precession, nutation and frame bias. Methods for computing the location of the CIP and the CIO have been given by Capitaine & Wallace (2003).

The coefficients of the equivalent bilinear rotation are

$$a = \exp\left(\mathrm{i}\frac{s}{2}\right), \quad b = \frac{X+\mathrm{i}Y}{1+Z}\exp\left(\mathrm{i}\frac{s}{2}\right) \tag{39}$$

which emphasizes, once again, the compactness and simplicity produced by using bilinear rotations. Imposing the unimodularity condition (8) gives

$$a = \sqrt{\frac{1+Z}{2}}\exp\left(\mathrm{i}\frac{s}{2}\right), \quad b = \frac{X+\mathrm{i}Y}{\sqrt{2(1+Z)}}\exp\left(\mathrm{i}\frac{s}{2}\right). \tag{40}$$

5 ABERRATION

The aberration of light due to the relative motion of the observer with respect to the fundamental reference frame shifts the apparent position of an object on the celestial sphere along a great circle toward the direction of motion by an amount given by

$$\tan\frac{\theta'}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{\theta}{2} = \mathrm{e}^{-\phi}\tan\frac{\theta}{2}, \tag{41}$$

where θ, θ' are true and apparent angles, respectively, c is the speed of light, v is the magnitude of the observer's velocity in the fundamental reference frame and the quantity ϕ is the rapidity.

It can be shown (Stuart 1984) that the angular separation, θ, between two points on the sphere with images under stereographic projection z and z_1 is given by

$$\tan\frac{\theta}{2} = \left|\frac{z-z_1}{\bar{z}_1 z+1}\right|.$$

Let z_1 be the complex number that is the image of the point on the celestial sphere representing the observer's direction of motion with respect to the rest frame. The bilinear rotation

$$\mathrm{T}_1(z) = \frac{z-z_1}{\bar{z}_1 z+1}$$

places z_1 at the origin of the complex plane and it follows immediately that in this coordinate system the effect of aberration amounts to a simple scaling by the factor given in equation (41). Moreover, in the original coordinate system, the aberration of light is described by the bilinear transformation

$$\mathrm{T}_1^{-1}\left(\mathrm{e}^{-\phi}.\mathrm{T}_1(z)\right) = \frac{\left(\mathrm{e}^{-(\phi/2)} + |z_1|^2\mathrm{e}^{(\phi/2)}\right)z + 2z_1\sinh(\phi/2)}{2z\bar{z}_1\sinh(\phi/2) + \left(\mathrm{e}^{(\phi/2)} + \mathrm{e}^{-(\phi/2)}|z_1|^2\right)}. \tag{42}$$

In equation (42) the coefficients of the bilinear transformation are normalized to $ad - bc = \left(1+|z_1|^2\right)^2$. Equation (42) applies a Lorentz boost in the direction of z_1 in an exact relativistically correct manner. It is thus a simple matter to incorporate aberration together with coordinate rotations in a compact and consistent framework. Furthermore, equation (42) shows that a pure Lorentz boost, with no associated rotation, is realized by a bilinear transformation (5) in which the coefficients a, d are real numbers and $b = \bar{c}$.

It is shown in Appendix A that with normalization $ad - bc = 1$ the bilinear coefficients equation (42) can be written as

$$\mathrm{T}_1^{-1}\left(\mathrm{e}^{-\phi}.\mathrm{T}_1(z)\right) = \frac{\left(\cosh(\phi/2) + Z_1\sinh(\phi/2)\right)z + (X_1+\mathrm{i}Y_1)\sinh(\phi/2)}{z(X_1-\mathrm{i}Y_1)\sinh(\phi/2) + \left(\cosh(\phi/2) - Z_1\sinh(\phi/2)\right)}, \tag{43}$$

where (X_1, Y_1, Z_1) is a unit vector in the direction of the boost.

The quantities $\mathrm{e}^{-\phi}$ and z_1 are most conveniently obtained using rectangular coordinates by means of the relations

$$v = \sqrt{\dot{X}^2+\dot{Y}^2+\dot{Z}^2}, \quad z_1 = \frac{\dot{X}\pm\mathrm{i}\dot{Y}}{v\mp\dot{Z}}, \quad \mathrm{e}^{-\phi} = \sqrt{\frac{c-v}{c+v}}, \tag{44}$$

where the dot denotes differentiation with respect to time and $c = 173.14463348\ \mathrm{au\,d^{-1}}$. The upper sign applies for stereographic projection in the form (2) and the lower for (9).

The bilinear transformation (42) can be written in the equivalent form

$$\mathrm{T}_1^{-1}\left(\mathrm{e}^{-\phi}.\mathrm{T}_1(z)\right) = \frac{\left(\mathrm{e}^{-\phi}+|z_1|^2\right)z + z_1\left(1-\mathrm{e}^{-\phi}\right)}{\bar{z}_1\left(1-\mathrm{e}^{-\phi}\right)z + \left(1+\mathrm{e}^{-\phi}|z_1|^2\right)} \tag{45}$$

which offers some computational efficiencies.

6 CONCLUSIONS

The foregoing paper has described the practical application of complex analysis to computations of the mean and apparent places of celestial objects. The methods described here place the correction for the aberration of light and coordinate rotations accounting for precession and nutation into a common and compact framework that is straightforward to implement in practice.

Detailed algorithms for computing the mean and apparent places of celestial objects have been given by Kaplan et al. (1989). Their methods can be used to determine light travel time and gravitational deflection of light. Having done this, the object's observed direction is known and can represented as a single complex number, z. The effect of aberration of light is obtained by performing the bilinear transformation (42). Precession and nutation are accounted for by applying bilinear rotations of the form (5), sequentially to the resulting complex number, with coefficients a and b given in Sections 3.1 and 3.2, respectively, or combined as in Section 4. Corrections for frame bias are applied similarly if needed. The final complex number result is then converted to spherical coordinates by means of equations (3) or rectangular coordinates using equations (4). Transformations specified in terms of combined frame bias–precession–nutation transformations are handled similarly.

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APPENDIX A: BILINEAR AND LORENTZ TRANSFORMATIONS

In units in which the speed of light $c = 1$, a general Lorentz transformation can be written in 2×2 complex matrix form as

$$\mathbf{M}' = \mathbf{L} \cdot \mathbf{M} \cdot \mathbf{L}^{\dagger}, \tag{A1}$$

where

$$\mathbf{M} = \begin{pmatrix} t+z & x+\mathrm{i}y \\ x-\mathrm{i}y & t-z \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A2}$$

with the constraint that the complex numbers a, b, c and d satisfy $ad - bc = 1$. $\mathbf{L}^{\dagger}$ is the Hermitian conjugate (transposed complex conjugate) of $\mathbf{L}$.

To establish the connection between the Lorentz transformation of points on the celestial sphere and the bilinear transformation of points on the complex plane, note that a light ray reaching the observer at $t = 0$ passes through space–time points satisfying the condition

$$|\mathbf{M}| = t^2 - x^2 - y^2 - z^2 = 0, \tag{A3}$$

where $|\mathbf{M}|$ denotes the determinant of $\mathbf{M}$. The 2×2 complex matrix of the form

$$\mathbf{W} = \begin{pmatrix} |w|^2 & w \\ \bar{w} & 1 \end{pmatrix} \tag{A4}$$

uniquely associates each light-like trajectory reaching the observer at $t = 0$ with a complex number, w. At the observer's coordinate time $t = 1$ a photon that passed through the observer's position at $t = 0$ will be located at a point (x, y, z) somewhere on the unit sphere, $x^2 + y^2 + z^2 = 1$. The image of this point under stereographic projection on to the complex plane is $w = \mathbf{M}_{12}/\mathbf{M}_{22} = (x + \mathrm{i}y)/(1 - z)$. Substituting $\mathbf{M} = \mathbf{W}$ in equation (A1) gives

$$\mathbf{M}'_{12}/\mathbf{M}'_{22} = \frac{aw + b}{cw + d} \tag{A5}$$

and it follows that in the Lorentz transformed frame, coincident with the original at $t = t' = 0$, the photon passes through a point at $t' = 1$ whose stereographic projection is obtained by applying the bilinear transformation (5).

The analysis of Section 5 exposes the close connection between rotations and Lorentz boosts. Renormalizing the coefficients of the bilinear transformation (42) such that $ad - bc = 1$ yields

$$a = \cosh\frac{\phi}{2} + \frac{|z_1|^2 - 1}{|z_1|^2 + 1}\sinh\frac{\phi}{2}, \tag{A6}$$

$$b = \bar{c} = \frac{2\mathrm{Re}z_1}{|z_1|^2 + 1}\sinh\frac{\phi}{2} + \mathrm{i}\frac{2\mathrm{Im}z_1}{|z_1|^2 + 1}\sinh\frac{\phi}{2}, \tag{A7}$$

$$d = \cosh\frac{\phi}{2} - \frac{|z_1|^2 - 1}{|z_1|^2 + 1}\sinh\frac{\phi}{2}. \tag{A8}$$

These should be compared to expressions given by Stuart (1984) for the coefficients of a bilinear rotation representing a right-handed rotation by an angle θ about an axis in the direction z_1. After applying the unimodularity condition (8) the results are

$$a = \bar{d} = \cos\frac{\theta}{2} - \mathrm{i}\frac{|z_1|^2 - 1}{|z_1|^2 + 1}\sin\frac{\theta}{2}, \tag{A9}$$

$$b = -\bar{c} = \frac{2\mathrm{Im}z_1}{|z_1|^2 + 1}\sin\frac{\theta}{2} - \mathrm{i}\frac{2\mathrm{Re}z_1}{|z_1|^2 + 1}\sin\frac{\theta}{2}. \tag{A10}$$

The relations (A9)–(A10) are transformed into (A6)–(A8) by setting $\theta \to \mathrm{i}\,\phi$. Thus a pure Lorentz boost in the direction defined by z_1 is equivalent to a rotation through an imaginary angle about an axis lying in the direction of the boost.

With the aid of equations (4) the coefficients of the bilinear transformation, (A9)–(A10), can be used to construct a vector

$$\frac{1}{a+d}\,(\mathrm{i}b + \mathrm{i}c,\ b - c,\ \mathrm{i}a - \mathrm{i}d) = (X,\ Y,\ Z)\tan\frac{\theta}{2} \tag{A11}$$

known as the Gibbs vector in which the unit vector (X, Y, Z) is the axis and θ is the angle of rotation. The right-hand side exhibits a singularity at $\theta = 180^\circ$ but methods exist to move it to other locations (Schaub & Junkins 1996).

The construction (A11) performed with equations (A6)–(A8) gives

$$\frac{1}{a+d}\,(\mathrm{i}b + \mathrm{i}c,\ b - c,\ \mathrm{i}a - \mathrm{i}d) = \mathrm{i}\,(X,\ Y,\ Z)\tan\frac{\phi}{2} \tag{A12}$$

and produces the analogue of the Gibbs vector for a Lorentz boost. In this case the unit vector (X, Y, Z) gives the direction of the boost and ϕ is the rapidity. Equation (43) can be obtained from this result.

A general Lorentz transformation can be constructed from rotations $\mathbf{R},\ \mathbf{R}'$ and boosts $\mathbf{B},\ \mathbf{B}'$ as

$$\mathbf{L} = \mathbf{B}\cdot\mathbf{R} = \mathbf{R}'\cdot\mathbf{B}'. \tag{A13}$$

As noted in Section 5, the matrix elements for rotations satisfy $c = -\bar{b},\ d = a$ and for a boost a, b are real and $c = \bar{d}$. From this it follows that $\mathbf{R}^\dagger = \mathbf{R}^{-1}$ and $\mathbf{B}^\dagger = \mathbf{B}$. Given a 2×2 matrix or bilinear transformation representing a Lorentz transformation the boosts, $\mathbf{B},\ \mathbf{B}'$ and thence the rotations $\mathbf{R},\ \mathbf{R}'$ can be extracted by noting that $\mathbf{L}\cdot\mathbf{L}^\dagger = \mathbf{B}^2$ and $\mathbf{L}^\dagger\cdot\mathbf{L} = \mathbf{B}'^2$.

As stated in the Introduction, several different methods exist to carry out coordinate rotations. The present work has focused on relating the standard 3D matrix rotations to bilinear rotations of points on the complex plane. To go in the opposite direction, let a and b be the coefficients for a bilinear rotation (7) satisfying the unimodularity condition (8). The equivalent 3D rotation matrix is

$$\mathbf{R} = \begin{bmatrix} \mathrm{Re}\left(a^2 - b^2\right) & -\mathrm{Im}\left(a^2 + b^2\right) & -2\mathrm{Re}\,(ab) \\ \mathrm{Im}\left(a^2 - b^2\right) & \mathrm{Re}\left(a^2 + b^2\right) & -2\mathrm{Im}\,(ab) \\ 2\mathrm{Re}\,(\bar{a}b) & 2\mathrm{Im}\,(\bar{a}b) & |a|^2 - |b|^2 \end{bmatrix}. \tag{A14}$$

In some applications, rotations are performed by exploiting the algebra of quaternions

$$\begin{aligned}(\mathrm{i}x' + \mathrm{j}y' + \mathrm{k}z') = {} & (q_0 + \mathrm{i}q_1 + \mathrm{j}q_2 + \mathrm{k}q_3) \\ & \cdot(\mathrm{i}x + \mathrm{j}y + \mathrm{k}z) \\ & \cdot(q_0 - \mathrm{i}q_1 - \mathrm{j}q_2 - \mathrm{k}q_3)\end{aligned} \tag{A15}$$

for real numbers q_i satisfying $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. The quantities $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are non-commuting under multiplication and satisfy the fundamental equations, $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$. The bilinear rotation coefficients are related to the quaternion components by $a = q_0 + \mathrm{i}\,q_3$ and $b = -q_2 + \mathrm{i}q_i$. Moreover, the quaternion components are related to the Gibbs vector (A11) by

$$(q_0 + \mathrm{i}q_1 + \mathrm{j}q_2 + \mathrm{k}q_3) = \cos\frac{\theta}{2} + (\mathrm{i}X + \mathrm{j}Y + \mathrm{k}Z)\sin\frac{\theta}{2}. \tag{A16}$$

The treatment of a general Lorentz transformation in a similar fashion requires the use of biquaternions (Lanczos 1949) and is not considered here.

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